Lemmas on the floor and ceiling functions

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Abstract

A few simple lemmas on the floor and ceiling functions.

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This note contains simples lemmas on the floor and ceiling functions, and on switching from one function to the other. These lemmas are not in the book Graham et al. (1989), but they could very well be added to it, maybe as slightly easy exercises. We have done a bibliographic search without success, but it is possible that they are already in literature. In which case, we would be happy to recieve an email with the correct reference.

Notations: For two integers a and b, we denote $a \div b$ the integer quotient of a divided by b; and we denote $a \bmod^+ b$ the "higher modulus" with $a \bmod^+ b = b$ if $a \bmod b = 0$, and $a \bmod^+ b = a \bmod b$ if $a \bmod b > 0$. It is possible that the "higher modulus" already exists in literature. Here again, we would be happy to recieve an email with the correct reference.

Let's start with some observations that are already in the folklore: if $k \in \mathbb{N}$, $\lceil \frac{k}{2} \rceil = \lfloor \frac{k+1}{2} \rfloor$ (Graham et al. (1989) for example), or $\lceil k \times \frac{2}{3} \rceil = \lfloor (k+1) \times \frac{2}{3} \rfloor$ (OEIS Foundation Inc. (2025)). Apparently, nobody took the time to generalise these folklore observations, but we see that:

Lemma 0.1. *Let* $k, a \in \mathbb{N}, a \geq 1$, *then* $\lceil k \times \frac{a-1}{a} \rceil = \lfloor (k+1) \times \frac{a-1}{a} \rfloor$.
Note that $\frac{a-1}{a} = 1 - \frac{1}{a}$.

Proof:

Idea: The gap $(k+1) \times \frac{a-1}{a} - k \times \frac{a-1}{a} = \frac{a-1}{a} < 1$. Hence if these two values are on both sides of an integer b, then $b = \lceil k \times \frac{a-1}{a} \rceil = \lfloor (k+1) \times \frac{a-1}{a} \rfloor$. Formal proof: $\lceil k \times \frac{a-1}{a} \rceil = \lceil k \times (1 - \frac{1}{a}) \rceil = \lceil k - \frac{k}{a} \rceil = k + \lceil -\frac{k}{a} \rceil = k - \lfloor \frac{k}{a} \rfloor$ and $\lfloor (k+1) \times \frac{a-1}{a} \rfloor = \lfloor (k+1) \times (1 - \frac{1}{a}) \rfloor = \lfloor (k+1) - \frac{k+1}{a} \rfloor = k + 1 + \lfloor -\frac{k+1}{a} \rfloor = k + 1 - \lceil \frac{k+1}{a} \rceil$. Let $q = k \div a$. We have $k - \lfloor \frac{k}{a} \rfloor = k - q$ and $k + 1 - \lceil \frac{k+1}{a} \rceil = k + 1 - (q+1) = k - q$. ■

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Note that it cannot be generalised further. Some examples of blocking: $\lceil 1\times\frac{1}{3}\rceil > \lfloor 2\times\frac{1}{3}\rfloor$, $\lceil 2\times\frac{3}{5}\rceil > \lfloor 3\times\frac{3}{5}\rfloor$, $\lceil 0\times\frac{3}{5}\rceil < \lfloor 2\times\frac{3}{5}\rfloor$, $\lceil 3\times\frac{3}{5}\rceil < \lfloor 5\times\frac{3}{5}\rfloor$, $\lceil 3\times\frac{5}{7}\rceil > \lfloor 4\times\frac{5}{7}\rfloor$, $\lceil 0\times\frac{5}{7}\rceil < \lfloor 2\times\frac{5}{7}\rfloor$, $\lceil 1\times\frac{5}{7}\rceil < \lfloor 3\times\frac{5}{7}\rfloor$. A last example that shows that k must be an integer: $\lceil \frac{a}{3\times(a-1)}\times\frac{a-1}{a}\rceil = 1 \neq \lfloor(\frac{a}{3\times(a-1)}+1)\times\frac{a-1}{a}\rfloor = \lfloor\frac{1}{3}+\frac{a-1}{a}\rfloor = 0$.

Note that we have also the known lemma (see Graham et al. (1989)):

Lemma 0.2. *Let $k,a\in\mathbb{N}, a\geq 1$, then $\lceil k\times\frac{1}{a}\rceil = \lfloor(k+a-1)\times\frac{1}{a}\rfloor$.*

Lemma 0.3. *Let $k,l,a\in\mathbb{N}, a\geq 2$. If $m = k+\lceil\frac{k}{a-1}\rceil$ and $M = k+\lceil\frac{k+1}{a-1}\rceil$, then $\lfloor l\times\frac{a-1}{a}\rfloor = k \Leftrightarrow m\leq l\leq M$.*

Proof:

Let $m = k+\lceil\frac{k}{a-1}\rceil$.
We have $\lfloor m\times\frac{a-1}{a}\rfloor = \lfloor(k+\lceil\frac{k}{a-1}\rceil)\times\frac{a-1}{a}\rfloor = \lfloor(k+\frac{k}{a-1}+\frac{(a-1)-k \bmod^+ (a-1)}{a-1})\times\frac{a-1}{a}\rfloor = \lfloor k+\frac{(a-1)-k \bmod^+ (a-1)}{a}\rfloor = k$
and $\lfloor(m-1)\times\frac{a-1}{a}\rfloor = \lfloor(k+\lceil\frac{k}{a-1}\rceil-1)\times\frac{a-1}{a}\rfloor = \lfloor(k+\frac{k}{a-1}+\frac{(a-1)-k \bmod^+ (a-1)}{a-1}-1)\times\frac{a-1}{a}\rfloor = \lfloor k+\frac{(a-1)-k \bmod^+ (a-1)}{a}-\frac{a-1}{a}\rfloor = k-1$.
Let $M = k+\lceil\frac{k+1}{a-1}\rceil$.
We have $\lfloor M\times\frac{a-1}{a}\rfloor = \lfloor(k+\lceil\frac{k+1}{a-1}\rceil)\times\frac{a-1}{a}\rfloor = \lfloor(k+\frac{k+1}{a-1}+\frac{(a-1)-(k+1) \bmod^+ (a-1)}{a-1})\times\frac{a-1}{a}\rfloor = \lfloor k+\frac{1}{a}+\frac{(a-1)-(k+1) \bmod^+ (a-1)}{a}\rfloor = k$
and $\lfloor(M+1)\times\frac{a-1}{a}\rfloor = \lfloor(k+\lceil\frac{k+1}{a-1}\rceil+1)\times\frac{a-1}{a}\rfloor = \lfloor(k+\frac{k+1}{a-1}+\frac{(a-1)-(k+1) \bmod^+ (a-1)}{a-1}+1)\times\frac{a-1}{a}\rfloor = \lfloor k+\frac{1}{a}+\frac{(a-1)-(k+1) \bmod^+ (a-1)}{a}+\frac{a-1}{a}\rfloor = k+1$. ■

Lemma 0.4. *Let $k,l,a\in\mathbb{N}, a\geq 2$. If $m = k+\lfloor\frac{k-1}{a-1}\rfloor$ and $M = k+\lfloor\frac{k}{a-1}\rfloor$, then $\lceil l\times\frac{a-1}{a}\rceil = k \Leftrightarrow m\leq l\leq M$.*

Proof:

Let $m = k+\lfloor\frac{k-1}{a-1}\rfloor$.
We have $\lceil m\times\frac{a-1}{a}\rceil = \lceil(k+\lfloor\frac{k-1}{a-1}\rfloor)\times\frac{a-1}{a}\rceil = \lceil(k+\frac{k-1}{a-1}-\frac{(k-1) \bmod (a-1)}{a-1})\times\frac{a-1}{a}\rceil = \lceil k-\frac{1}{a}-\frac{(k-1) \bmod (a-1)}{a}\rceil = k$
and $\lceil(m-1)\times\frac{a-1}{a}\rceil = \lceil(k+\lfloor\frac{k-1}{a-1}\rfloor-1)\times\frac{a-1}{a}\rceil = \lceil(k+\frac{k-1}{a-1}-\frac{(k-1) \bmod (a-1)}{a-1}-1)\times\frac{a-1}{a}\rceil = \lceil k-\frac{1}{a}-\frac{(k-1) \bmod (a-1)}{a}-\frac{a-1}{a}\rceil = k-1$.
Let $M = k+\lfloor\frac{k}{a-1}\rfloor$.
We have $\lceil M\times\frac{a-1}{a}\rceil = \lceil(k+\lfloor\frac{k}{a-1}\rfloor)\times\frac{a-1}{a}\rceil = \lceil(k+\frac{k}{a-1}-\frac{k \bmod (a-1)}{a-1})\times\frac{a-1}{a}\rceil = \lceil k-\frac{k \bmod (a-1)}{a}\rceil = k$
and $\lceil(M+1)\times\frac{a-1}{a}\rceil = \lceil(k+\lfloor\frac{k}{a-1}\rfloor+1)\times\frac{a-1}{a}\rceil = \lceil(k+\frac{k}{a-1}-\frac{k \bmod (a-1)}{a-1}+1)\times\frac{a-1}{a}\rceil = \lceil k-\frac{k \bmod (a-1)}{a}+\frac{a-1}{a}\rceil = k+1$. ■

These integer parts are quite frequent in combinatorics, for exemple $k\rightarrow\lceil k\times\frac{a-1}{a}\rceil$ corresponds to a recursive cutting of a set of size n where the largest part keeps some fraction of the elements of the set rounded to the integer above. This cutting must stop when $n = a - 1$. We can then count the number of steps to reach $a - 1$, *i.e.* the depth of the tree of the cuts. If we want to generate the sequence of the number of steps by increasing order of the values of n, the bound M above is very useful. Here is an example of pseudo-code of a very efficient generator:

```
// Input a or a-1
b = a-1;
n = b;
M = b;
k = 0;
while(true){
  while(n <= M){
    yield k;
    ++n;
  }
  M += M/b;
  ++k;
}
/*
Example with a = 3
n 2 3 4 5 6 7 8 9
M 2 3 4 6 6 9 9 9
k 0 1 2 3 3 4 4 4
*/
```

The same reasoning applies to a recursive cutting of a set of size n where the largest part keeps some fraction of the elements of the set rounded to the integer below. This cutting must stop when $n = 0$. The pseudo-code of the generator is slightly less efficient:

```
// Input a ou a-1
b = a-1;
n = 0;
M = 0;
k = 0;
while(true){
  while(n <= M){
    yield k;
    ++n;
  }
  M += (M+b)/b;
  ++k;
}
```

```
/*
Example with a = 3
n 0 1 2 3 4 5 6 7
M 0 1 2 4 4 7 7 7
k 0 1 2 3 3 4 4 4
*/
```

For a week, I make additions to the OEIS, and in particular sequence A061420. And the editors don't want that we are too verbose in details, in explanations or in generalisations; hence this article is kind of a way to evade censorship. But I understand that we cannot have a full article for each sequence of the OEIS.

Thanks God! Thanks Father! Thanks Jesus! Thanks Holy-Spirit!

References

R. L. Graham, D. E. Knuth, and O. Patashnik. *Concrete Mathematics: A Foundation for Computer Science*. Addison-Wesley, Reading, 1989.

OEIS Foundation Inc. (2025). Entry A061420 in The On-Line Encyclopedia of Integer Sequences. URL https://oeis.org/A061420.

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